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Non-perturbative aspects of chiral anomalies

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Abstract

We investigate the properties of chiral anomalies in $d = 2$ in the framework of constructive quantum field theory. The condition that the gauge propagator is sufficiently soft in the ultraviolet is essential for the anomaly non-renormalization; when it is violated, as for contact current–current interactions, the anomaly is renormalized by higher order corrections. The same conditions are also essential for the validity, in the massless case of the closed equation obtained combining Ward identities and Schwinger–Dyson equations; this solves the apparent contradiction between perturbative computations and exact analysis.

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1. Introduction

Anomalies are the breaking of certain classical symmetries happening in quantum field theory (QFT) (see [A, BJ]). A well-known example is in QED_4, in which the axial Ward identity (WI) is

$$\mathbf{p}_\mu \hat{\Gamma}_5^\mu(\mathbf{p},\mathbf{k}) = \gamma^5(\hat{S}(\mathbf{k}-\mathbf{p}))^{-1} - \gamma^5(\hat{S}(\mathbf{k}))^{-1} + 2\mathrm{i}m\hat{\Gamma}_5(\mathbf{p},\mathbf{k}) + \mathrm{i}\alpha\hat{F}(\mathbf{p},\mathbf{k}), \tag{1.1}$$

where $\Gamma_5^\mu(\mathbf{p},\mathbf{k})$ and $\hat{\Gamma}_5(\mathbf{p},\mathbf{k})$ are the Fourier transform of $\langle T(j^5_{\mu,\mathbf{z}}\bar{\psi}_\mathbf{x}\psi_\mathbf{y})\rangle_A$ and $\langle T(j^5_\mathbf{z}\bar{\psi}_\mathbf{x}\psi_\mathbf{y})\rangle_A$ (A means truncation with respect to the fermion interacting propagator $\hat{S}(\mathbf{k})$), and $\hat{F}(\mathbf{p},\mathbf{k})$ is the Fourier transform of $\varepsilon_{\mu,\nu,\rho,\sigma}\langle T(F_{\mu,\nu,\mathbf{z}}F_{\rho,\sigma,\mathbf{z}}\bar{\psi}_\mathbf{x}\psi_\mathbf{y})\rangle_A$. The last term, which is unexpected from formal applications of the classical Noether theorem, is the anomaly; it was shown by Adler [A] that α is exactly quadratic in the charge $\alpha = \frac{e^2}{16\pi^2}$, that is the anomaly is *non-renormalized* by higher order corrections. An accurate derivation of this property, known as *Adler–Bardeen theorem*, was given in [AB], in which (1.1) with $\alpha = \frac{e^2}{16\pi^2}$ was proved as a *perturbative order by order* identity among formal expansions in Feynman graphs. In most textbooks, the properties of the anomalies are actually derived following the functional integral approach in [F], in which however the gauge fields are treated as external classical fields, so that higher order corrections to the anomaly would be in any case neglected. On the other hand several objections have been raised against the validity of the anomaly non-renormalization along the years, starting from [JJ] (see [A1] for a recent review), so that a non-perturbative derivation of it would be highly desirable, in view also of the role of such a property in the proof of renormalizability of the electroweak model. This is actually far from the present possibilities in $d = 4$, so that anomalies have been investigated in $d = 2$, with the hope of getting results beyond a purely perturbative level and to have insights for the $d = 4$ case.

In [GR] it was shown by a formal expansion in Feynman graphs that the anomaly non-renormalization holds also for $d = 2$ QFT models *either for massive gauge or for Thirring interactions*. The advantage of the $d = 2$ case is that one can use the operatorial exact solutions to get an 'explicit verification of the perturbation-theory calculations'. Indeed, following the analysis in [J], the anomaly non-renormalization appears as a consequence of the validity of the Ward identities for the total and axial current and of the Schwinger–Dyson equation; such equations can be combined in a *closed equation* for the two and four point functions and from a self-consistency argument the explicit value of the anomalies is obtained, showing the absence of higher order corrections.

Note that the validity of the anomaly non-renormalization for Thirring interactions does not follow from the Adler–Bardeen theorem in $d = 2$, as there the fast decay of the bosonic propagator plays an essential role. Indeed other perturbative computations [AF] have shown that in the $d = 2$ massless Thirring model there are higher orders contribution to the anomaly. Note also that the question on whether or not one can use exact results to infer properties about the correlations computed in a functional integral approach is not trivial at all and it was the subject of extensive debates (see for instance [GL]).

The recent developments in the mathematical analysis of quantum models at low dimension make it finally possible to investigate the properties of the chiral anomalies at a non-perturbative level in the framework of constructive QFT. In such an approach, the Euclidean n-point functions are obtained as the limit of functional integrals suitably regularized through lattice or momentum cut-offs; Feynman graph expansions are avoided for their bad combinatorial properties, and cluster expansions are instead used, which allow us to prove the convergence of the series involved. While a well-known problem in this approach is posed by the basic conflict between the scale decompositions used in a non-perturbative setting [P, G] and the local symmetries, the methods recently developed in [BM] overcome such a problem, at least in $d = 2$, and allow the rigorous construction of QFT models in $d = 2$ showing that the momentum cut-offs can be removed and that the resulting Schwinger functions verify the axioms. By using such methods it has been rigorously proved in [M, BFM] that the condition that the gauge propagator is sufficiently soft in the ultraviolet is *essential* for the anomaly non-renormalization in a functional integral approach; when it is violated, as for Thirring current–current interactions, the anomaly is *renormalized* by higher order corrections. Such results confirm, at a non-perturbative level, the perturbative analysis in [AB] in which the decay of the boson propagator plays an essential role; they are however in apparent contrast with the results based on the exact solutions in which the anomaly non-renormalization seems not to require such conditions.

In this paper we will explain how to resolve such apparent contradiction, and we finally clarify the relation between exact analysis and functional approach in $d = 2$ models. By combining the WI with the Schwinger–Dyson equation, at *finite* cut-offs, one does not obtain a closed equation as there are additional corrections depending on a complicate way from the cut-offs. We will prove that such corrections are indeed vanishing when the cut-offs are removed *provided that* the same conditions ensuring the anomaly non-renormalization are verified. Such conditions require that the boson propagator decays fast enough for large momenta, or at least that the boson cut-off is removed *after* the fermionic one; in purely fermionic models, it is necessary to start with *non-local* current–current interactions taking the local limit *after* the removal of the fermionic cut-off. If such conditions are not verified, as for Thirring contact interactions, the corrections are not vanishing so that the closed equation postulated by the exact solutions is *not verified* by the correlations computed from functional integrals; this solves the apparent contradictions between the functional integral approach and exact analysis.

The paper is organized in the following way. In section 2 the main results are presented; in sections 3 and 4 we construct the model and study the anomalies, referring for the complete proofs, which are quite long and technical, to [M, BFM]. Finally in section 5 we present some new results, analyzing the corrections deriving the closed equations for the two-point functions .

2. Main results

2.1. The model

We consider the (Euclidean) $d = 2$ QFT model whose correlations can be obtained from the *generating function*

$$\mathcal{W}_{K,N}(J,\phi) = \log\int P_{Z_2}(\mathrm{d}\psi^{(\leqslant N)})P(\mathrm{d}A) \times \exp\left(\left(\int \mathrm{d}\mathbf{x}\,[eZ_1\bar\psi_{\mathbf{x}}(A_{\mu,\mathbf{x}}\gamma_\mu)\psi_{\mathbf{x}} + J_{\mu,\mathbf{x}}A_{\mu,\mathbf{x}} + \phi_{\mathbf{x}}\bar\psi_{\mathbf{x}} + \bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}]\right)\right) \tag{2.1}$$

$\phi_{\mathbf{x}}, \bar\phi_{\mathbf{x}}, J_{\mu,\mathbf{x}}$ are external fields, Z_2 is the fermionic wavefunction renormalizations, Z_1 is the charge renormalization, $\psi, \bar\psi$ are Grassmann variables and $P_{Z_2}(\mathrm{d}\psi^{(\leqslant N)})$ is the Grassmannian integration with propagator

$$g^{(\leqslant N)}(\mathbf{x}-\mathbf{y}) = \frac{1}{Z_2}\int \mathrm{d}\mathbf{p}\frac{-\mathrm{i}\,\not{\mathbf{p}} + Z_4 m}{\mathbf{p}^2 + Z_4^2 m^2}\,\mathrm{e}^{-\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{y})}\chi_N(\mathbf{p}), \tag{2.2}$$

where $\chi_N(\mathbf{k})$ is a *smooth cut-off function* non-vanishing for $|\mathbf{k}| \leqslant 2^{N+1}$ and $= 1$ for $|\mathbf{k}| \leqslant \gamma^N$, N being a positive integer. Finally $A_{\mu,\mathbf{x}} = (A_{0,\mathbf{x}}, A_{1,\mathbf{x}})$ are Euclidean boson fields with Gaussian measure $P(dA)$ and the propagator $\langle A_{\mu,\mathbf{x}}A_{\nu,\mathbf{y}}\rangle = \delta_{\mu,\nu}v_K(\mathbf{x}-\mathbf{y})$; if A_μ is a massive vector field its covariance is

$$v_K(\mathbf{x}-\mathbf{y}) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{y})}\frac{\chi_K(\mathbf{p})}{\mathbf{p}^2 + M^2} \tag{2.3}$$

but we will mostly consider the case

$$v_K(\mathbf{x}-\mathbf{y}) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{y})}\chi_K(\mathbf{p}). \tag{2.4}$$

The reason is that the theory with propagator (2.4) has a perturbative structure much more similar to $d = 4$ gauge models, as it is renormalizable with divergence index is $2 - \frac{f}{2} - b$, if b, f are the external bosonic and fermionic lines (to be compared with $4 - \frac{3}{2}f - b$ for QED_4), while with the choice (2.3) the theory is super-renormalizable and the index is $2 - n - f/2$, if n is the perturbative order.

The truncated Euclidean Schwinger functions are defined as

$$\langle\psi_{\mathbf{x}_1}\ldots\psi_{\mathbf{x}_n}\bar\psi_{\mathbf{y}_1}\ldots\bar\psi_{\mathbf{y}_n}A_{\mu_1,\mathbf{z}_1}\ldots A_{\mu_m,\mathbf{z}_m}\rangle_{K,N} = \left.\frac{\partial^{2n+m}\mathcal{W}_{K,N}(J^A,\phi)}{\partial\phi_{\mathbf{x}_1}\ldots\partial\phi_{\mathbf{x}_n}\partial\bar\phi_{\mathbf{y}_1}\ldots\partial\bar\phi_{\mathbf{y}_n}\partial J^A_{\mu_1,\mathbf{z}_1}\ldots\partial J^A_{\mu_m,\mathbf{z}_m}}\right|_0. \tag{2.5}$$

We remark that the Schwinger functions (2.5) *cannot be* explicitly computed, even when $m = 0$, unless some approximation is done, as in [FGS], which is equivalent to treating the gauge field as a classical field; indeed the model (2.1) is strictly related to certain non-solvable statistical mechanics models describing $d = 2$ Ising models coupled by a quartic spin interaction, see [GM1].

By integrating the boson field a purely fermionic theory is obtained:

$$e^{\mathcal{W}_{K,N}(J,\phi)} = \int P_{Z_2}(\mathrm{d}\psi^{(\leqslant N)}) \exp\left(\frac{e^2}{2}\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,v_K(\mathbf{x}-\mathbf{y})[Z_1 e\bar\psi_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}} + J_{\mu,\mathbf{x}}] \right. \\ \left. \times [Z_1 e\bar\psi_{\mathbf{y}}\gamma_\mu\psi_{\mathbf{y}} + J_{\mu,\mathbf{y}}] + \int \mathrm{d}\mathbf{x}\,[\phi_{\mathbf{x}}\bar\psi_{\mathbf{x}} + \bar\phi_{\mathbf{x}}\psi_{\mathbf{x}}]\right). \tag{2.6}$$

The cut-offs make the functional integral (2.1) well defined; to carry out the renormalization program at a *non-perturbative* level we have to prove that it is possible to fix the bare parameters as functions of the ultraviolet cut-offs so that in the limit $K, N \to \infty$, the Schwinger functions exist and verify the Osterwalder–Schrader [OS] axioms. Different properties will be found, in the case (2.4), depending on whether the fermionic or the bosonic cut-off is removed first.

2.2. Removing the fermionic ultraviolet cut-off before the bosonic one

Let us consider first (2.1) with bosonic propagator (2.4) and assume the fermionic cut-off is removed first; this is equivalent to considering (2.6) assuming that the limit of local current–current interaction is performed after the removal of the fermionic cut-off.

We will prove that, if e is small enough (uniformly in m), by choosing the bare parameters as

$$Z_1 = Z_2 \equiv Z = \gamma^{-\eta K}, \qquad Z_4 = \gamma^{-\eta_1 K} \tag{2.7}$$

with η, η_1 being analytic functions of e and $\eta = ae^4 + O(e^6)$, $\eta_1 = be^2 + O(e^4)$, $a, b > 0$ suitable constants, the limit

$$\lim_{K\to\infty}\lim_{N\to\infty}\left\langle \psi_{\mathbf{x}_1}\ldots\psi_{\mathbf{x}_n}\bar\psi_{\mathbf{y}_1}\ldots\bar\psi_{\mathbf{y}_n}A_{\mu_1,\mathbf{z}_1}\ldots A_{\mu_m,\mathbf{z}_m}\right\rangle_{K,N} \tag{2.8}$$

exists at non-coinciding points and verifies the axioms.

By a chiral transformation the following axial Ward identity at finite cut-off is obtained

$$\mathbf{p}^\mu\left\langle j^5_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\right\rangle_{K,N} = \gamma^5\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} - \gamma^5\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle_{K,N} \\ + m\left\langle j^5_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\right\rangle_{K,N} + \alpha_{K,N}(\mathbf{p},\mathbf{k})\varepsilon_{\mu,\upsilon}\mathrm{i}\mathbf{p}_\mu\langle A_{\upsilon,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} \tag{2.9}$$

with $j^5_\mu = Z_2\bar\psi\gamma_\mu\gamma_5\psi$, $j^5 = Z_4\bar\psi\gamma_5\psi$, and

$$\lim_{K\to\infty}\lim_{N\to\infty}\alpha_{K,N}(\mathbf{p},\mathbf{k}) = \frac{e}{2\pi}. \tag{2.10}$$

As our results are uniform in the fermionic mass, we can write the *Schwinger–Dyson* equation in the massless case $m = 0$ ($Z \equiv Z_1$)

$$\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle_{K,N} = \chi_N(\mathbf{k})\frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2}\left[Z^{-1} - e^2\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\hat v_K(\mathbf{p})\gamma_\mu\langle j_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N}\right], \tag{2.11}$$

which can be combined with the WI (2.9) and its analogy for the current obtaining

$$\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle_{K,N} = \mathcal{B}_{1,K,N}(\mathbf{k}) + \frac{\chi_N(\mathbf{k})}{Z}\frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2} \\ - e^2\frac{[\bar a_1 - a_1]}{2}\chi_N(\mathbf{k})\left(\frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2}\right)\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\,\mathrm{i}\hat v_K(\mathbf{p})\frac{\gamma_\mu\mathbf{p}_\mu}{\mathbf{p}^2}\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N}, \tag{2.12}$$

where $a_1^{-1} = 1 - \frac{e^2}{2\pi}$ and $\bar{a}_1^{-1} = 1 + \frac{e^2}{2\pi}$, and $\mathcal{B}_{1,K,N}(\mathbf{k})$ is a term depending on the *integrated difference between the WI with or without cut-off*, and it is such that

$$\lim_{K\to\infty}\lim_{N\to\infty} \mathcal{B}_{1,K,N}(\mathbf{k}) = 0. \tag{2.13}$$

Finally, as a corollary of the above results, in the case (2.1) with bosonic propagator (2.3) keeping K finite, the bare parameters must be chosen as N-independent and (2.9), (2.10), (2.13) hold.

2.3. *Removing the bosonic ultraviolet cut-off before the fermionic one*

We consider now (2.1) with bosonic propagator (2.4) and we assume that the bosonic cut-off is removed first; this is equivalent to considering (2.6), assuming that the removal of the fermionic cut-off is done starting from a local current–current interaction.

If e is small enough (uniformly in m), by choosing the bare parameters as

$$Z_1 = Z_2 \equiv Z = \gamma^{-\eta N}, \qquad Z_4 = \gamma^{-\eta_1 N} \tag{2.14}$$

with η, η_1 being analytic functions of e and $\eta = ae^4 + O(e^6)$, $\eta_1 = be^2 + O(e^4)$, $a, b > 0$ suitable constants, the limit

$$\lim_{N\to\infty}\lim_{K\to\infty} \left\langle \psi_{\mathbf{x}_1}\ldots\psi_{\mathbf{x}_n}\bar{\psi}_{\mathbf{y}_1}\ldots\bar{\psi}_{\mathbf{y}_n}A_{\mu_1,\mathbf{z}_1}\ldots A_{\mu_m,\mathbf{z}_m}\right\rangle_{K,N} \tag{2.15}$$

exists at non-coinciding points and verifies the Osterwalder–Schroeder axioms [OS].

By a chiral transformation the following axial ward identity at finite cut-off is obtained:

$$\begin{aligned}&\mathbf{p}^\mu\left\langle j^5_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}-\mathbf{p}}\right\rangle_{K,N} = \gamma^5\langle\psi_{\mathbf{k}-\mathbf{p}}\bar{\psi}_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} - \\ &\gamma^5\langle\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}}\rangle_{K,N} + m\left\langle j^5_{\mathbf{p}}\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}-\mathbf{p}}\right\rangle_{K,N} + \alpha_{K,N}(\mathbf{p},\mathbf{k})\varepsilon_{\mu,\upsilon}\mathrm{i}\mathbf{p}_\mu\langle A_{\upsilon,\mathbf{p}}\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}-\mathbf{p}}\rangle_{K,N}\end{aligned} \tag{2.16}$$

$$\lim_{N\to\infty}\lim_{K\to\infty}\alpha_{K,N}(\mathbf{p},\mathbf{k}) = \frac{e}{2\pi} + Ae^3 + O(e^4) \tag{2.17}$$

with $A > 0$ being a non-vanishing constant (see (4.8) below).

By combining the Schwinger–Dyson equation (2.11) with the WI (2.16) and its analogy for the current at finite cut-off we obtain

$$\begin{aligned}\langle\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}}\rangle_{K,N} &= \mathcal{B}_{2,K,N}(\mathbf{k}) + \frac{\chi_N(\mathbf{k})}{Z}\frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2} - e^2\frac{[\bar{a}_2 - a_2]}{2}\chi_N(\mathbf{k})\left(\frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2}\right) \\ &\quad\times\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\,\mathrm{i}\hat{v}_K(\mathbf{p})\frac{\gamma_\mu\mathbf{p}_\mu}{\mathbf{p}^2}\langle\psi_{\mathbf{k}-\mathbf{p}}\bar{\psi}_{\mathbf{k}-\mathbf{p}}\rangle_{K,N}\end{aligned} \tag{2.18}$$

and $a_2^{-1} = 1 - \frac{e^2}{2\pi} - Ae^4 + O(e^6)$, $\bar{a}_2^{-1} = 1 + \frac{e^2}{2\pi} - Ae^4 + O(e^6)$; again $\mathcal{B}_{2,K,N}(\mathbf{k})$ is a term depending on the integrated difference between the WI with or without cut-off, but in this case it is not vanishing at all, but it holds

$$\mathcal{B}_{2,K,N}(\mathbf{k}) = \sigma\chi_N(\mathbf{k})\left(\frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2}\right) + \rho\langle\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}}\rangle + H_{K,N}(\mathbf{k}) \tag{2.19}$$

with $\rho = \bar{A}e^4 + O(e^6)$, with $\bar{A} > 0$, $\sigma = O(e^6)$ and $\lim_{N\to\infty}\lim_{K\to\infty}H_{K,N}(\mathbf{k}) = 0$.

2.4. *Remarks.*

(1) Note that, according to power counting, there are *four* marginal or relevant monomials (namely $\bar{\psi}\psi$, $j^\mu j^\mu$, $A^\mu j^\mu$, $\bar{\psi}\not{\partial}\psi$), but all the ultraviolet divergences can be reabsorbed in *two* bare parameters (Z, Z_4) only, as a consequence of the Ward identities.

(2) The anomaly non-renormalization is true if the bosonic propagator is given by (2.3). In contrast, in the theory with bosonic propagator (2.4) the validity of the anomaly non-renormalization depends on the order in which the cut-offs are removed. The anomaly is not renormalized if the fermionic cut-off is removed first (see (2.10)), while the anomaly has higher order corrections if the bosonic cut-off is removed first (see (2.17)). In the purely fermionic model (2.6), the anomaly non-renormalization is found only starting from a non-local current–current interaction, while if one starts from a local interaction higher order corrections are found. As the anomaly non-renormalization has several important physical applications, for instance the anomaly concellations in the electroweak model, the above results suggest that regularizations have to be chosen properly in order to ensure the validity of such a property.

(3) The same conditions ensuring the validity of the anomaly non-renormalization ensures the validity of a closed equation for the two-point Schwinger function, in the limit in which cut-offs are removed, which is equal to the one *formally* obtained by *first* removing the cut-offs and *then* combining the WI with the Schwinger–Dyson equation (see (2.13)). In contrast, if the bosonic cut-off is removed first the two-point function verifies a *different* closed equation, as a consequence of (2.19).

(4) The above fact explains the apparent contradiction between the exact analysis of the Thirring model and the functional integral approach. In the exact analysis one starts assuming the validity of a certain closed equation, which is verified in the functional integral approach, only starting from a non-local current–current interaction, and performing the local limit *after* the removal of the fermionic cut-off. In contrast, if one starts from a local interaction, the closed equation is different from the one postulated in the exact approach.

3. Removal of cut-offs and construction of the theory

We consider for definiteness the massless case $m = 0$ and we set the fermionic external field equal to zero $\phi = 0$ (the inclusion of the mass and fermionic external field is straightforward). We consider (2.4), (2.6) first assuming that *the fermionic cut-off is larger than the bosonic one* $N \geqslant K$, and calling $\lambda = e^2/2$ we write

$$\int P_Z(\mathrm{d}\psi^{(\leqslant N)})\,\exp\left(\lambda\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\,v_K(\mathbf{x}-\mathbf{y})\big(\bar\psi_{\mathbf{x}}^{(\leqslant N)}\gamma_\mu\psi_{\mathbf{x}}^{(\leqslant N)}\big)\big(\bar\psi_{\mathbf{y}}^{(\leqslant N)}\gamma_\mu\psi_{\mathbf{y}}^{(\leqslant N)}\big)\right.$$
$$\left.+\int \mathrm{d}\mathbf{x}\,J_{\mu,\mathbf{x}}\bar\psi_{\mathbf{x}}^{(\leqslant N)}\gamma_\mu\psi_{\mathbf{x}}^{(\leqslant N)}\right)$$
$$\equiv \int P_Z(\mathrm{d}\psi^{(\leqslant N)})\,\mathrm{e}^{\mathcal{V}^{(N)}(\sqrt{Z_1}\psi^{(\leqslant N)},J)}. \tag{3.1}$$

We can represent the fermionic propagator $\hat g^{(\leqslant N)}(\mathbf{k}) = \frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2}\frac{\chi_N(\mathbf{k})}{Z}$ as

$$\hat g^{(\leqslant N)}(\mathbf{k}) = \hat g^{(\leqslant N-1)}(\mathbf{k}) + \hat g^{(N)}(\mathbf{k}), \tag{3.2}$$

where

$$\hat g^{(N)}(\mathbf{k}) = \frac{1}{Z}\frac{-\mathrm{i}\not{\mathbf{k}}}{\mathbf{k}^2}f^{(N)}(\mathbf{k}), \qquad \chi_N(\mathbf{k}) = \chi_{N-1}(\mathbf{k}) + f^{(N)}(\mathbf{k})$$

and $f^{(N)}(\mathbf{k})$ is a smooth function with support in $2^{N-1} \leqslant |\mathbf{k}| \leqslant 2^{N+1}$. By the properties of Grassman integrals (3.1) can be rewritten as

$$\int P_Z(\mathrm{d}\psi^{(\leqslant N-1)})\int P_Z(\mathrm{d}\psi^{(N)})\,\mathrm{e}^{\mathcal{V}^{(N)}(\psi^{(\leqslant N-1)}+\psi^{(N)},J)} = \int P_Z(\mathrm{d}\psi^{(\leqslant N-1)})\,\mathrm{e}^{\mathcal{V}^{(N-1)}(\psi^{(\leqslant N-1)},J)}, \tag{3.3}$$

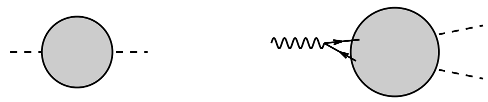

Figure 1. Graphical representation of $H_{2,0}^{(N-1)}$ and $H_{2,1}^{(N-1)}$; the dotted half-lines represent the external fermionic fields and the wiggly half-line represents J.

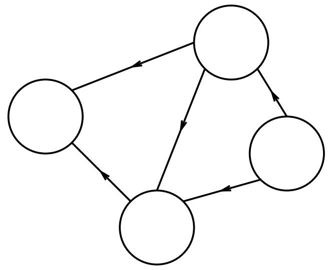

Figure 2. An example of Feynman graph contributing to $\mathcal{E}_N^T$: the clusters of points $P_1, \ldots, P_s$ are represented as circles.

where $P_Z(\mathrm{d}\psi^{(N)})$ has the propagator $g^{(N)}(\mathbf{x})$ and $\mathcal{V}^{(N-1)}(\psi^{(\leqslant N-1)}, J)$ has the form (see figure 1)

$$\mathcal{V}^{(N-1)}(\psi, J) = \sum_{\substack{n,m\\ n+m\geqslant 0}} H_{2n,m}^{(N-1)}(\underline{\mathbf{x}}, \underline{\mathbf{y}}; \underline{\mathbf{z}}) \prod_{i=1}^{m} J_{\mathbf{z}_i} \prod_{i=1}^{n} \psi_{\mathbf{x}_i}^{(\leqslant N-1)} \bar{\psi}_{\mathbf{y}_i}^{(\leqslant N-1)}, \tag{3.4}$$

where $H_{2n,m}^{(N-1)}$ are given by

$$\begin{aligned} H_{2n,m}^{(N-1)}(\underline{\mathbf{z}}; \underline{\mathbf{x}}, \underline{\mathbf{y}}) = \frac{1}{m!}\frac{1}{2n!}\left[\prod_{i=1}^{m}\frac{\partial}{\partial J_{\mathbf{z}_i}}\right]\left[\prod_{i=1}^{n}\frac{\partial}{\partial \psi_{\mathbf{x}_i}^{(\leqslant N-1)}}\frac{\partial}{\partial \bar{\psi}_{\mathbf{y}_i}^{(\leqslant N-1)}}\right] \\ \sum_{k=1}^{\infty}\frac{1}{k!}\mathcal{E}_N^T(\mathcal{V}^{(N)}(\psi^{(\leqslant N-1)} + \psi^{(N)}, J)\ldots\mathcal{V}^{(N)}(\psi^{(\leqslant N-1)} + \psi^{(N)}, J))\Big|_{\psi^{(N)}=J=0} \end{aligned} \tag{3.5}$$

and $\mathcal{E}_N^T$ is the *truncated expectation* with respect to the propagator $g^{(N)}(\mathbf{x})$.

The truncated expectation is a linear operation defined starting from monomials in the following way. If $\tilde{\psi}^{(N)}(P)$ are monomials in the fields, that is, if $P_+ \bigcup P_- = P$

$$\tilde{\psi}^{(N)}(P) = \prod_{f\in P_-} \psi_{\mathbf{x}(f)}^{(N)} \prod_{f\in P_+} \bar{\psi}_{\mathbf{y}(f)}^{(N)} \tag{3.6}$$

the truncated expectation $\mathcal{E}_N^T(\tilde{\psi}(P_1)\ldots\tilde{\psi}(P_s))$ is given by the sum of the values (with the relative sign) of all possible connected Feynman graphs, obtained representing the monomial $\tilde{\psi}(P)$ as a number of oriented half-lines coming out from a cluster of points and contracting them in all possible ways so that all the clusters are connected; to each line is associated a propagator $g^{(N)}$ (see figure 2).

This implies that the kernels $H_{2n,m}^{(N-1)}$ can be written as a sum over Feynman graphs as well, and the presence of cut-offs makes each of them finite. However even if each Feynman graph, if no J fields are present, is bounded by $C^k|\lambda|^k/k!$, their number is $O(k!^2)$ so that each term in (3.5) is bounded by $C^k|\lambda|^k k!$ from which convergence does not follow. Such $k!$-factorial bounds are generally obtained either for bosonic or fermionic theories. On the other hand,

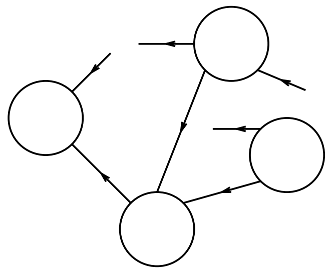

Figure 3. Graphical representation of one term in (3.7); the propagators in T are represented as lines while the fields corresponding to the propagators in the determinant are represented as unpaired half-lines.

in the case of fermions, anticommutativity produces dramatic cancellations among Feynman graphs (which are lost if the sum of graphs is simply bounded by the sum of their absolute values) and the result is that a *regularized* fermionic perturbation series converges, see [C].

In order to exploit such cancellations it is convenient to use a different representation of the truncated expectations (see figure 3) [GK] (for a tutorial derivation, see also [GM]).

$$\mathcal{E}_N^T(\tilde{\psi}^{(N)}(P_1), \ldots, \tilde{\psi}^{(N)}(P_s)) = \sum_T \prod_{l\in T} g^{(N)}(\mathbf{x}_l - \mathbf{y}_l) \int \mathrm{d}P_T(\mathbf{t}) \det G^{N,T}(\mathbf{t}), \tag{3.7}$$

where T is a set of lines forming an *anchored tree graph* between the clusters of points $\mathbf{x}^{(i)} \cup \mathbf{y}^{(i)}$, that is T is a set of lines, which becomes a tree graph if one identifies all the points in the same cluster. Moreover, $\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \leqslant i, i' \leqslant s\}, \mathrm{d}P_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in R^s$ of unit norm. Finally $G^{h,T}(\mathbf{t})$ is a $\left(\sum_{i=1}^s |P_i|/2 - s + 1\right) \times \left(\sum_{i=1}^s |P_i|/2 - s + 1\right)$ matrix, whose elements are given by $G^{N,T}_{ij,i'j'} = t_{i,i'} g^{(N)}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'})$ with $\left(f^-_{ij}, f^+_{i'j'}\right)$ not belonging to $T\left(f^{\pm}_{i'j'} \in P^{\pm}_j\right)$.

Note that $\det G^{N,T}(\mathbf{t})$ is a *Gram determinant* which can be bounded by, for a suitable constant C

$$|\det G^{N,T}(\mathbf{t})| \leqslant C^{(\sum_{i=1}^s |P_i|/2 - s + 1)N} \gamma^{(\sum_{i=1}^s |P_i|/2 - s + 1)N} \tag{3.8}$$

and no factorials are present in the above bound; in contrast, if one expands the determinant (obtaining essentially the Feynman graph expansion) and bounds each term with its modulus, one gets essentially a similar bound times an extra factorial. Another important point is that the sum over the trees T is bounded by $C^s s!$.

By bounding each term in the expansion (3.5) by (3.7), (3.8), it is not difficult to see that the series in (3.5) is convergent for λ small enough. We can then integrate the fields $\psi^{(N-2)}, \ldots, \psi^{(h)}$ with respect to the fermionic integration $P(\psi^{(k)}), K \leqslant k \leqslant N$ with propagator $g^{(k)}(\mathbf{x})$ verifying the bound, for any M (by integration by parts):

$$|g^{(k)}(\mathbf{x})| \leqslant C_M \frac{1}{Z} \frac{\gamma^k}{1 + (\gamma^k |\mathbf{x}|)^M}. \tag{3.9}$$

Again $\mathcal{V}^{(k)}(\psi^{(\leqslant k)}, J)$ can be written as in (3.4) (with k replacing $N-1$), and using the analogous of (3.7), (3.8), the bound $\int \mathrm{d}\mathbf{r} |g^{(h)}(\mathbf{r})| \leqslant C\gamma^{-h}$ and that, for any M

$$|v_K(\mathbf{x})| \leqslant C_M \frac{\gamma^{2K}}{1 + (\gamma^K |\mathbf{x}|)^M} \tag{3.10}$$

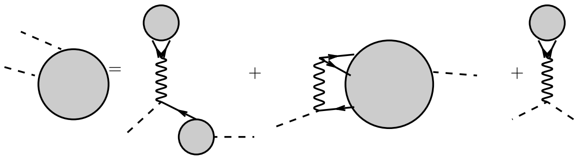

Figure 4. Graphical representation of (3.13); the blobs represent $H^{(h)}_{n,m}$ as in figure 1, the paired wiggly lines represent v_K, the paired line $g^{(h,N)}$.

we find, if $||f|| = \frac{1}{L^2}\int \mathrm{d}\underline{\mathbf{r}}|f(\underline{\mathbf{r}})|$ and $|\lambda| \leqslant \varepsilon$, the following bound

$$\left\| H^{(h)}_{n,m} \right\| \leqslant C\varepsilon^{\max(0,\frac{n}{2}-1)}\gamma^{h(2-\frac{n}{2}-m)}. \tag{3.11}$$

(3.11) is called a *power counting* bound; it says that the kind of bound one expects from naive *dimensional* considerations is valid also at a *non-perturbative* level. It is however not sufficient for taking the limit $N \to \infty$, as the *scaling dimension* $2-\frac{n}{2}-m$ is *non-negative*; one have then to improve the bound for $n = 2$, $m = 0$, 1 or $n = 0$, $m = 2$ or $n = 4$, $m = 0$ (in all other cases the dimension is negative). Of course the improvement must be done somewhat respecting the determinant structure in the truncated expectations, in order to avoid combinatorial problems.

We use the following property of truncated expectations

$$\begin{aligned}\mathcal{E}^T(\tilde\psi(P_1\cup P_2)\tilde\psi(P_3)\dots\tilde\psi(P_n)) &= \mathcal{E}^T(\tilde\psi(P_1)\tilde\psi(P_2)\dots\tilde\psi(P_n))\\ &+\sum_{\substack{K_1,K_2,K_1/K_2=0\\ K_1\cup K_2=(3,\dots,n)=\{\alpha_i\}_{i=1}^{|K_1|+|K_2|}}}\mathcal{E}^T(\tilde\psi(P_1)\tilde\psi(P_{\alpha_i})\dots\tilde\psi(P_{\alpha_{|K_1|}})\mathcal{E}^T\\ &\times\left(\tilde\psi(P_2)\tilde\psi\left(P_{\alpha_{|K_1|+1}}\right)\dots\tilde\psi\left(P_{\alpha_{|K_1|+|K_2|}}\right)\right).\end{aligned} \tag{3.12}$$

Note that the number of terms in the sum in the rhs of (3.12) is bounded by C^n for a suitable constant C. The above equation has a very simple meaning. The truncated expectation is given by a sum of two kinds of graphs: the first is such that cutting the connection between P_1 and P_2 the graph is still connected, and the second is such that is disconnected under such operation.

From the analogous of (3.5), with $\mathcal{E}^T_N$ replaced by $\mathcal{E}^T_{h+1,N}$ with propagator $g^{(h,N)}(\mathbf{x}-\mathbf{y}) = \sum_{k=h}^N g^{(k)}(\mathbf{x}-\mathbf{y})$ and using (3.12) (see figure 4)

$$\begin{aligned}H^{(h)}_{2,0}(\mathbf{x},\mathbf{y}) &= \int \mathrm{d}\mathbf{y}_1\lambda v_K(\mathbf{x}-\mathbf{y}_1)H^{(h)}_{0,1}(\mathbf{y}_1)g^{(h,N)}(\mathbf{x}-\mathbf{y}_2)H^{(h)}_{2,0}(\mathbf{y}_2;\mathbf{y})\\ &+\lambda\int \mathrm{d}\mathbf{y}_2 v_K(\mathbf{x}-\mathbf{y}_1)g^{(h,N)}(\mathbf{x}-\mathbf{y}_2)H^{(h)}_{2,1}(\mathbf{y},\mathbf{y}_2;\mathbf{y}_1)\\ &+\lambda\delta(\mathbf{x}-\mathbf{y})\int \mathrm{d}\mathbf{y}_1 v_K(\mathbf{x}-\mathbf{y}_1)H^{(h)}_{0,1}(\mathbf{y}_1).\end{aligned} \tag{3.13}$$

In the massless case $m = 0$ the first and the third terms are vanishing; hence, using that $\|g^{(j)}\|_1 \leqslant \tilde C\gamma^{-j}$ and $\|v\|_\infty \leqslant C\gamma^{2K}$, we obtain the following bound

$$\left\|H^{(h)}_{2,0}\right\| \leqslant |\lambda|\cdot\|v\|_\infty\cdot\left\|H^{(h)}_{2,1}\right\|\cdot\sum_{j=h}^N\|g^{(j)}\|_1 \leqslant \frac{\tilde C}{1-\gamma^{-1}}C_0|\lambda|\gamma^h\gamma^{-2h+2K}. \tag{3.14}$$

Note that we have a gain $O(\gamma^{-2(h-K)})$ with respect to the bound (3.11), due to the fact that we are integrating over a fermionic instead of that over a bosonic line.

Similar arguments can be repeated for $H^{(h)}_{0,2}$, which can be decomposed as in the following picture.

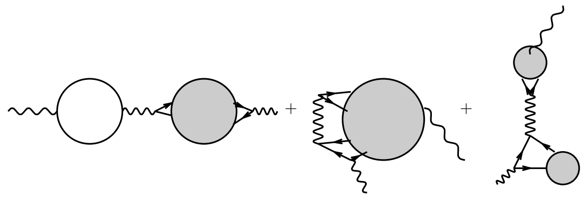

Figure 5. Decomposition of $H_{2,0}^{(h)}$: the blobs represent $H_{n,m}^{(h)}$, the paired wiggly lines represent v_K, the paired line $g^{(h,N)}$.

The second term in figure 5 is given by

$$\lambda\int \mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{u}'\,\mathrm{d}\mathbf{z}'\,\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{w}' v(\mathbf{u}'-\mathbf{z}')g^{(h,N)}(\mathbf{w}-\mathbf{u})g^{(h,N)}(\mathbf{w}-\mathbf{u}')g^{(h,N)}(\mathbf{u}'-\mathbf{w}')H_{2,2}^{(h)}(\mathbf{w}',\mathbf{u};\mathbf{z},\mathbf{z}'). \tag{3.15}$$

It is convenient to decompose the three propagators into scales, $\sum_{j,i,i'=h}^{N} g^{(j)}g^{(i)}g^{(i')}$ and then, for any realization of j, i, i', to take the $\|\cdot\|_1$ norm on the two propagators on the higher scales, and the $\|\cdot\|_\infty$ norm on the propagator with the lowest one. In this way we can bound (3.15), using (3.11), by

$$\begin{aligned}|\lambda||v|_\infty\cdot\left\|H_{2;2}^{(h)}\right\|3!\sum_{j=h}^{N}\sum_{i=h}^{j}\sum_{i'=h}^{i}\|g^{(j)}\|_1\|g^{(i)}\|_1\|g^{(i')}\|_\infty &\leqslant C_1|\lambda|\gamma^{2K}\gamma^{-h}\sum_{j=h}^{N}\sum_{i=h}^{j}\sum_{i'=h}^{i}\gamma^{-j}\gamma^{-\mathrm{i}}\gamma^{i'}\\ &\leqslant C_2|\lambda|\gamma^{2K}\gamma^{-h}\sum_{j=h}^{N}\gamma^{-j}(j-h)\leqslant C_2|\lambda|\gamma^{2K}\gamma^{-h}\sum_{j=h}^{N}\gamma^{-j}\gamma^{(j-h)/2}\\ &\leqslant C_4|\lambda|\gamma^{-2(h-K)}.\end{aligned} \tag{3.16}$$

A similar bound is found for the third term in figure 5; regarding the first term, we can rewrite it as

$$\begin{aligned}\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,[g^{(h,N)}(\mathbf{z}-\mathbf{x})]^2\lambda v_K(\mathbf{x}-\bar{\mathbf{z}})H_{0,2}^{(h)}(\bar{\mathbf{z}},\mathbf{y}) &= \int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,\lambda v_K(\bar{\mathbf{z}}-\mathbf{z})[g^{(h,N)}(\mathbf{x}-\mathbf{z})]^2 H_{0,2}^{(h)}(\bar{\mathbf{z}},\mathbf{y})\\ &+\int \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}\,[v_K(\bar{\mathbf{z}}-\mathbf{x})-v_K(\bar{\mathbf{z}}-\mathbf{z})][g^{(h,N)}(\mathbf{x}-\mathbf{z})]^2\lambda v_K(\mathbf{z}-\mathbf{z}')H_{0,2}^{(h)}(\bar{\mathbf{z}},\mathbf{y})\end{aligned} \tag{3.17}$$

and using that

$$\int \mathrm{d}\mathbf{x}\,[g^{(h,N)}(\mathbf{x}-\mathbf{z})]^2 = 0 \tag{3.18}$$

we see that the first term in (3.17) is vanishing; on the other hand the difference $[v_K(\bar{\mathbf{z}}-\mathbf{x})-v_K(\bar{\mathbf{z}}-\mathbf{z})]$ produces an extra factor γ^{K-h} in the bounds.

As similar analysis can be repeated also for $H_{4,0}$, $H_{2,1}$ and in all cases an extra factor γ^{-h+K}, with respect to the dimensional bounds, is obtained, so that we can integrate the fields $\psi^{(N)}, \psi^{(N-1)}, \ldots, \psi^{(K)}$ (the *ultraviolet* regime) obtaining a sequence of kernels $\mathcal{V}^{(N)}, \ldots, \mathcal{V}^{(K)}$ whose kernels are well defined in the $N\to\infty$ limit.

The integration of the fields with scale $K-1, K-2, \ldots$ (the *infrared regime*) has to be done in a different way ($\gamma^{K-h}\geqslant 1$ in such case), defined iteratively in the following way. Assume that we have integrate all fields up to scale $h<K$ obtaining

$$\int P_{Z_h}(\mathrm{d}\psi^{(\leqslant h)})\,\mathrm{e}^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)},J)}, \tag{3.19}$$

where $P_{Z_h}(\mathrm{d}\psi^{(\leqslant h)})$ has a propagator similar to (2.2) with χ_N replaced by χ_h, Z replaced by Z_h and $\mathcal{V}^h(\psi)$ of the form (3.4).

We write $\mathcal{V}^{(h)}$ as

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}, J) = \mathcal{L}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}, J) + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}, J), \tag{3.20}$$

where $\mathcal{R} = 1 - \mathcal{L}$ and $\mathcal{L}$ is a linear operation acting on $H^{(h)}_{n,m}$ in the following way:

$$\begin{aligned} &\mathcal{L}H^{(h)}_{2,1}(\underline{\mathbf{k}}) = H^{(h)}_{2,1}(0) \qquad H^{(h)}_{4,0}(\underline{\mathbf{k}}) = H^{(h)}_{4,0}(0) \\ &\mathcal{L}H^{(h)}_{2,0}(\underline{\mathbf{k}}) = H^{(h)}_{2,0}(0) + \mathbf{k}_\mu \partial_\mu H^{(h)}_{2,0}(0) \end{aligned} \tag{3.21}$$

an by parity $H^{(h)}_{2,0}(0) = 0$. We can include the quadratic part in the free fermionic interaction obtaining

$$\int P_{Z_{h-1}}(\mathrm{d}\psi^{(\leqslant h-1)}) \int P_{Z_{h-1}}(\mathrm{d}\psi^{(h)}) \exp\bigg(-\lambda_h Z^2_{h-1} \int \mathrm{d}\mathbf{x}\big(\psi^{(\leqslant h)}_{\mathbf{x}} \gamma_\mu \psi^{(\leqslant h)}_{\mathbf{x}}\big)^2 \\ + Z^{(1)}_{h-1} \int \mathrm{d}\mathbf{x} J_{\mu,\mathbf{x}} \psi^{(\leqslant h)}_{\mathbf{x}} \gamma_\mu \psi^{(\leqslant h)}_{\mathbf{x}} + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}, J)\bigg), \tag{3.22}$$

and integrating the field ψ^h we return to an expression like (3.19) and the procedure can be iterated. The result of this procedure is that the n-point functions are expressed by expansions in the effective couplings $\lambda_h, \ldots, \lambda_K, \lambda$ which are *convergent* if $\sup_{h\leqslant j\leqslant N} |\lambda_h|$ is small enough. The extraction of the local part produces an improvement in the size of the kernels, producing derivatives applied on the external fields, giving an extra γ^h, and factors $(\mathbf{x}-\mathbf{y})$, if $\mathbf{x}$, $\mathbf{y}$ are the coordinate of the external fields; this last factor can be bounded using that $\gamma^h \int \mathrm{d}\mathbf{z}|\mathbf{z}||\mathrm{g}^{(\mathrm{h})}(\mathbf{z})| \leqslant \mathrm{C}\gamma^{-\mathrm{h}}$ or $\int \mathrm{d}\mathbf{z}|\mathbf{z}||\mathrm{v}(\mathbf{z})| \leqslant \mathrm{C}\gamma^{-\mathrm{K}} \leqslant \mathrm{C}\gamma^{-\mathrm{h}}$ (this last inequality explains why the procedure used in (3.19) cannot be used for scales $\geqslant K$). The major problem is to show that λ_h and $\frac{Z^{(1)}_h}{Z_h}$ remain close to their initial value. This was proved in [BM], by combining ward identities and Schwinger–Dyson equations at each integration step, and the consequence of this analysis is the relations

$$\lambda_h = \lambda + O(\lambda^2), \qquad Z_h = \gamma^{-\eta h}(1 + O(\lambda)), \qquad \frac{Z^{(1)}_h}{Z_h} = (1 + O(\lambda)), \tag{3.23}$$

As $Z = Z_N(1 + O(\lambda))$, with the choice (2.7) we can remove first the fermionic cut-off $N \to \infty$ and then the bosonic one $K \to \infty$. A similar analysis can be repeated in the case of the propagator (2.4) with fixed K.

Let us discuss now shortly what happens when $K \geqslant N$; now there is no gain in integrating over a fermionic instead of a bosonic line, as $\gamma^{K-h} \geqslant 1$ for any h; the integration of the scales $N, N-1, \ldots$ is done following for all the momentum scales the integration procedure (3.19), as in such a case $\int \mathrm{d}\mathbf{z}|\mathbf{z}||\mathrm{v}(\mathbf{z})| \leqslant \mathrm{C}\gamma^{-\mathrm{K}} \leqslant \mathrm{C}\gamma^{-\mathrm{h}}$ for any h. The result is that, by choosing the bare parameters as in (3.23) with $h = N$, the limit $N \to \infty$ can be performed after the limit $K \to \infty$.

4. Renormalization or non-renormalization of the anomalies

Let us consider for simplicity again the case of massless fermions $m = 0$; we consider (2.4), (2.6) first assuming that *the fermionic cut-off is larger than the bosonic one* $N \geqslant K$ so that, performing in (2.6) the local phase transformation $\psi_{\mathbf{x}} \to \mathrm{e}^{\mathrm{i}\alpha^5_{\mathbf{x}}\gamma^5}\psi_{\mathbf{x}}$ and $\bar\psi_{\mathbf{x}} \to \bar\psi_{\mathbf{x}}\mathrm{e}^{\mathrm{i}\alpha^5_{\mathbf{x}}\gamma^5}$ we find

$$\mathbf{p}_\mu\big\langle j^5_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\big\rangle_{K,N} = \gamma^5\langle\psi_{\mathbf{k}-\mathbf{p}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} - \gamma^5\langle\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}}\rangle_{K,N} + \big\langle\delta j^5_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\big\rangle_{K,N}, \tag{4.1}$$

where $j_{\mathbf{p}}^{\mu,5} = Z\int\frac{d\mathbf{k}'}{(2\pi)^2}\bar\psi_{\mathbf{k}'}\gamma_\mu\gamma_5\psi_{\mathbf{k}'-\mathbf{p}}$ and $\delta j_{\mathbf{p}}^5 = Z\int\frac{d\mathbf{k}'}{(2\pi)^2}C_\mu(\mathbf{k}',\mathbf{k}'-\mathbf{p})\bar\psi_{\mathbf{k}'}\gamma_\mu\gamma_5\psi_{\mathbf{k}'-\mathbf{p}}$ with

$$C_\mu(\mathbf{k}_-,\mathbf{k}_+) = \left(\chi_N^{-1}(\mathbf{k}_-)-1\right)\mathbf{k}_{-,\mu} - \left(\chi_N^{-1}(\mathbf{k}_+)-1\right)\mathbf{k}_{+,\mu}. \tag{4.2}$$

The presence of the last term in (4.1) is due to the presence of the momentum fermionic cut-off which breaks local invariance. An analogous WI for the current is found performing the transformation $\bar\psi_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}$ and $\bar\psi_{\mathbf{x}} \to \bar\psi_{\mathbf{x}}e^{-i\alpha_{\mathbf{x}}}$. Note that

$$e\hat v_K(\mathbf{p})\left\langle j_{\mathbf{p},\mu}^5\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\right\rangle = i\varepsilon_{\mu,\nu}\langle A_{\nu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle. \tag{4.3}$$

Writing $\psi_{\mathbf{x}} = (\psi_{\mathbf{x},+},\psi_{\mathbf{x},-})$ and introducing an index $\omega=\pm$ denoting the chirality of the fermions we can write

$$\left\langle\delta j_{\mathbf{p}}^5\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\right\rangle_{K,N} = \frac{\lambda}{\pi e}\varepsilon_{\mu,\nu}\, i\mathbf{p}_\mu\langle A_{\nu,\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle_{K,N} + \sum_{\omega=\pm}\omega R_{\omega,K,N}(\mathbf{k},\mathbf{p}), \tag{4.4}$$

where

$$R_{\omega,K,N}(\mathbf{k},\mathbf{p}) = \frac{\partial^3}{\partial\bar J_{\mathbf{p}}\partial\phi_{\mathbf{k}}\partial\bar\phi_{\mathbf{k}-\mathbf{p}}}\mathcal{W}_{\omega,K,N}(\bar J,\phi)|_{0,0} \tag{4.5}$$

with

$$\begin{aligned} e^{\mathcal{W}_{\omega,K,N}(J,\phi)} &= \int P_Z(d\psi^{(\leqslant N)})\exp\left(\lambda(Z)^2\int\frac{d\mathbf{k}_1}{(2\pi)^2}\frac{d\mathbf{k}_2}{(2\pi)^2}v_K(\mathbf{p})\left(\bar\psi_{\mathbf{k}_1}\gamma_\mu\psi_{\mathbf{k}_1-\mathbf{p}}\right)\left(\bar\psi_{\mathbf{k}_2}\gamma_\mu\psi_{\mathbf{k}_2+\mathbf{p}}\right)\right)\\ &\times\exp\left(\int\frac{d\mathbf{k}}{(2\pi)^2}[\bar\psi_{\mathbf{k}}\phi_{\mathbf{k}}+\psi_{\mathbf{k}}\bar\phi_{\mathbf{k}}]\right)\\ &\times\exp\left(\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{d\mathbf{k}'}{(2\pi)^2}\bar J_{\mathbf{p}}Z\left[C_\omega(\mathbf{k}',\mathbf{k}'-\mathbf{p})\psi^+_{\omega,\mathbf{k}'}\psi_{\omega,\mathbf{k}'-\mathbf{p}}\right.\right.\\ &\left.\left.-\nu_- D_{-\omega}(\mathbf{p})\hat v_K(\mathbf{p})\psi^+_{-\omega,\mathbf{k}'}\psi_{-\omega,\mathbf{k}'-\mathbf{p}}\right],\right) \end{aligned} \tag{4.6}$$

where

$$D_\omega(\mathbf{p}) = -ip_0+\omega p \tag{4.7}$$

$\nu_- = \frac{\lambda}{\pi}$ and

$$C_\omega(\mathbf{k},\mathbf{k}-\mathbf{p}) = D_\omega(\mathbf{k}-\mathbf{p})\left(\chi_N^{-1}(\mathbf{k}-\mathbf{p})-1\right) - \left(\chi_N^{-1}(\mathbf{k})-1\right)D_\omega(\mathbf{k}). \tag{4.8}$$

Note that the functional integral (4.6) is similar to (3.1), the only difference being that the term $\int\frac{d\mathbf{p}}{(2\pi)^2}\frac{d\mathbf{k}'}{(2\pi)^2}\bar J_{\mathbf{p}}Z\psi^+_{\omega,\mathbf{k}'}\psi_{\omega,\mathbf{k}'-\mathbf{p}}$ in (3.1) is replaced by the exponent in the second line of (4.6), which is given by the sum of the two terms one of which is non-local. We can integrate (4.6) following an iterative procedure similar to that described in section 3, obtaining a sequence of $\tilde{\mathcal{V}}^{(h)}(\psi,\bar J)$ (the analogous of $\mathcal{V}^{(h)}(\psi,\bar J)$ for (3.1)) of the form

$$\tilde{\mathcal{V}}^{(h)}(\psi,\bar J) = \sum_{\substack{n,m\\ n+m\geqslant 0}} G^{(h)}_{2n,m}(\underline{\mathbf{z}};\underline{\mathbf{x}},\underline{\mathbf{y}})\prod_{i=1}^m\bar J_{\mathbf{z}_i}\prod_{i=1}^n\psi^{(\leqslant h)}_{\mathbf{x}_i}\bar\psi^{(\leqslant h)}_{\mathbf{y}_i}. \tag{4.9}$$

As in section 3, we need an improvement with respect to the dimensional bounds in the case of $G^{(h)}_{2,1}$ which has a vanishing scaling dimension. As there are two terms linear in $\bar J$ in the exponent (4.6), we can write (see figure 6)

$$G^{(h)}_{2,1} = G^{(h)}_{a,2,1} + G^{(h)}_{b,2,1}, \tag{4.10}$$

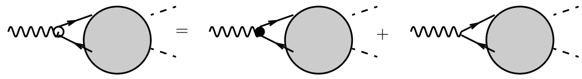

Figure 6. Graphical representation of (4.10); the wiggly line represents $\bar{J}$.

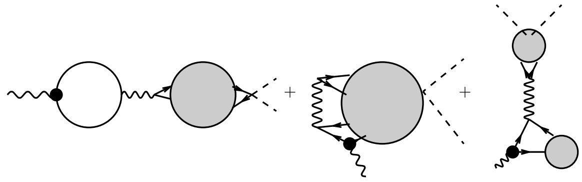

Figure 7. Decomposition for $G^{(h)}_{a,2,1}$; the gray blobs represent $H^{(h)}_{n,m}$, the paired lines the fermionic propagators and the wiggly lines the interactions.

where

$$G^{(h)}_{a,2,1} = \frac{1}{2}\frac{\partial}{\partial\psi^{(\leqslant h)}_{\mathbf{x}}}\frac{\partial}{\partial\bar{\psi}^{(\leqslant h)}_{\mathbf{y}}}$$

$$\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\mathcal{E}^T_{h+1,N}\left(Z\left[\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}C_\omega(\mathbf{k}',\mathbf{k}'-\mathbf{p})\psi^+_{\omega,\mathbf{k}'}\psi_{\omega,\mathbf{k}'-\mathbf{p}}\right]\mathcal{V}^{(N)}\ldots\mathcal{V}^{(N)}\right)\Bigg|_{\psi^{(\leqslant h)}=0} \tag{4.11}$$

and

$$G^{(h)}_{a,2,1} = \frac{1}{2}\frac{\partial}{\partial\psi^{(\leqslant h)}_{\mathbf{x}}}\frac{\partial}{\partial\bar{\psi}^{(\leqslant h)}_{\mathbf{y}}}$$

$$\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\mathcal{E}^T_{h+1,N}\left(Z\left[\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\nu_- D_{-\omega}(\mathbf{p})\hat{v}_K(\mathbf{p})\psi^+_{-\omega,\mathbf{k}'}\psi_{-\omega,\mathbf{k}'-\mathbf{p}}\right]\mathcal{V}^{(N)}\ldots\mathcal{V}^{(N)}\right)\Bigg|_{\psi^{(\leqslant h)}=0}. \tag{4.12}$$

When the two fields appearing in $\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\bar{J}_{\mathbf{p}}Z\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}C_\omega(\mathbf{k}',\mathbf{k}'-\mathbf{p})\psi^+_{\omega,\mathbf{k}'}\psi_{\omega,\mathbf{k}'-\mathbf{p}}$ are contracted one get

$$\Delta^{(h,k)}(\mathbf{k}^+,\mathbf{k}^-) = g^{(h)}(\mathbf{k}^+)C_\omega(\mathbf{k}^+,\mathbf{k}^-)g^{(k)}(\mathbf{k}^-) = (\mathbf{k}_+-\mathbf{k}_-)\cdot\mathbf{S}^{h,k}(\mathbf{k}_+,\mathbf{k}_-) \tag{4.13}$$

with the important property that

$$\Delta^{h,k}(\mathbf{k}^+,\mathbf{k}^-) = 0 \qquad h,k<N \tag{4.14}$$

and

$$|\mathbf{S}^{N,j}(\mathbf{z}-\mathbf{x},\mathbf{z}-\mathbf{y})| \leqslant C_M\frac{\gamma^N}{1+[\gamma^N|\mathbf{z}-\mathbf{x}|]^M}\frac{\gamma^j}{1+[\gamma^j|\mathbf{z}-\mathbf{y}|]^M}. \tag{4.15}$$

From (4.14) we see that at least one of the fields in $\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\bar{J}_{\mathbf{p}}Z\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}C_\omega(\mathbf{k}',\mathbf{k}'-\mathbf{p})\psi^+_{\omega,\mathbf{k}'}\psi^-_{\omega,\mathbf{k}'-\mathbf{p}}$ must be contracted at scale N.

We can decompose $G^{(h)}_{a,2,1}$ as explained in figure 7.

Regarding the second term, given by

$$\sum_{i,j=h}^{N}\int\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{u}'\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{w}'\mathbf{S}^{(i,j)}(\mathbf{z};\mathbf{u},\mathbf{w})g(\mathbf{u}-\mathbf{u}')v(\mathbf{u}-\mathbf{w}')H^{(h)}_{1,4}(\mathbf{w}';\mathbf{u}',\mathbf{w},\mathbf{x},\mathbf{y}) \tag{4.16}$$

we can proceed exactly as for (3.16), the main difference being that, from (4.13), either i or j has to be N, so that (4.16) is bounded by

$$C_1|\lambda|\gamma^{2K}\gamma^{-h}\sum_{i=h}^{N}\sum_{i'=h}^{i}\gamma^{-N}\gamma^{-i}\gamma^{i'} \leqslant C_2|\lambda|\gamma^{2K}\gamma^{-h-N}(N-h) \leqslant C_3|\lambda|\gamma^{-2(h-K)}\gamma^{-(1/2)(N-h)}. \tag{4.17}$$

A similar bound holds for the third term in figure 7; regarding the first term we can perform the analogous of the decomposition in (3.17) but now the fermionic bubble is not vanishing; such term is however exactly cancelled by $G^{(h)}_{b,2,1}$ provided that ν_- is chosen equal to

$$\begin{aligned}\nu_- &= 4\lambda\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\frac{C_{\omega,N}(\mathbf{k},\mathbf{k}-\mathbf{p})}{D_{-\omega}(\mathbf{p})}g^{(\leqslant N)}_{\omega}(\mathbf{k})g^{(\leqslant N)}_{\omega}(\mathbf{k}-\mathbf{p})\bigg|_{\mathbf{p}=0}\\ &= -\lambda\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\frac{k_0}{|\mathbf{k}|}\chi_0'(|\mathbf{k}|)D^{-1}_{\omega}(\mathbf{k}) = \frac{\lambda}{\pi}\int_0^\infty \mathrm{d}\rho\,\chi_0'(\rho) = \frac{\lambda}{\pi}.\end{aligned} \tag{4.18}$$

The conclusion is that, for $k \geqslant K$,

$$\left\|G^{(k)}_{2,1}\right\| \leqslant C|\lambda|\gamma^{\frac{1}{2}(k-N)} \tag{4.19}$$

and for $n \geqslant 4$

$$\left\|G^{(k)}_{n,1}\right\| \leqslant C|\lambda|^{\max(0,\frac{n}{2}-1)}\gamma^{k(1-\frac{n}{2})}\gamma^{\frac{1}{2}(k-N)}. \tag{4.20}$$

The consequence is that $R_{\omega,K,N}(\mathbf{k},\mathbf{p})$ verifies the same bound as $\langle A_{\mathbf{p}}\psi_{\mathbf{k}}\bar\psi_{\mathbf{k}-\mathbf{p}}\rangle$ time an extra factor $\gamma^{\frac{1}{2}(h_{\mathbf{k}}-N)}$, if $h_{\mathbf{k}}$ is the scale of $\mathbf{k}$, so that

$$\lim_{N\to\infty} R_{\omega,K,N}(\mathbf{k},\mathbf{p}) = 0. \tag{4.21}$$

A similar analysis can be repeated in the case of propagator (2.4) with fixed K. Note that the result of such an analysis is the anomaly is not renormalized by higher order corrections, and such a conclusion is obtained by completely avoiding any cancellation argument based on Feynman graph expansion, so that it is non-perturbative and perfectly rigorous.

Let us discuss now shortly what happens when $K \geqslant N$ still in the case (2.4), (2.6); in such a case there is no reason for which the contribution coming from the second and third addends of figure 7 should be small. Indeed we can proceed as in (3.19) and after the integration of the scales $N, N-1, \ldots, h$ we arrive at (setting again $\phi = 0$ for definiteness), if $\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)}, J)$ is the same as in (3.19)

$$\begin{aligned}&\int P_{Z_h}(\mathrm{d}\psi^{(\leqslant h)})\,\exp(-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leqslant h)},0) + \mathcal{R}\mathcal{B}^h(\psi^{(\leqslant h)},\bar J))\\ &\qquad\times\exp\bigg(\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\bar J_{\mathbf{p}}Z_h\big[C_\omega(\mathbf{k}',\mathbf{k}'-\mathbf{p})\psi^{+(\leqslant h)}_{\omega,\mathbf{k}'}\psi^{(\leqslant h)}_{\omega,\mathbf{k}'-\mathbf{p}}\\ &\qquad-\nu_{-,h}D_{-\omega}(\mathbf{p})\psi^{+(\leqslant h)}_{-\omega,\mathbf{k}'}\psi^{(\leqslant h)}_{-\omega,\mathbf{k}'-\mathbf{p}} - \nu_{+,h}D_\omega(\mathbf{p})\psi^{+(\leqslant h)}_{\omega,\mathbf{k}'}\psi^{(\leqslant h)}_{\omega,\mathbf{k}'-\mathbf{p}}\big]\bigg),\end{aligned} \tag{4.22}$$

where $\mathcal{B}^h(\psi^{(\leqslant h)},\bar J)$ is the sum of monomial with at least a $\bar J$-field and $\nu_{\pm,h} = O(\gamma^{\frac{1}{2}(h-N)}\lambda)$ provided that we choose

$$\nu_- = \frac{\lambda}{\pi} + O(\lambda^2) \qquad \nu_+ = A\lambda^2 + O(\lambda^3) \tag{4.23}$$

with

$$A = 4\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\left[\frac{u_0(|\mathbf{k}|)\chi_0(|\mathbf{k}|)}{|\mathbf{k}|^4} - \frac{\chi_0'(|\mathbf{k}|)}{2|\mathbf{k}|^3}\right]\int\frac{\mathrm{d}\mathbf{k}''}{(2\pi)^2}g^{(\leqslant N)}_{-\omega}(\mathbf{k}'')g^{(\leqslant N)}_{-\omega}(\mathbf{k}-\mathbf{k}'')D^2_{-\omega}(\mathbf{k}) > 0. \tag{4.24}$$

From the fact that $\nu_{\pm,h} = O(\gamma^{\frac{1}{2}(h-N)}\lambda)$ it follows that (4.21) holds also in this case. Note that (4.23) and (4.24) imply that in this case the anomaly is renormalized by higher order corrections.

5. Schwinger–Dyson equation

In this section we discuss the equation for the two-point Schwinger function one obtains combining the Schwinger–Dyson equation (2.12) with the WI. Again we start from the case (2.4), (2.6) assuming that the fermionic cut-off is larger than the bosonic one $N \geqslant K$ so that we find (2.12) with

$$\mathcal{B}_{1,K,N}(\mathbf{k}) = \frac{\chi_N(\mathbf{k})}{D_\omega(\mathbf{k})} \sum_{\varepsilon=\pm} \frac{a_1 - \varepsilon \bar{a}_1}{2} \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^2} \frac{\hat{v}_K(\mathbf{p})}{D_{-\omega}(\mathbf{p})} R_{\varepsilon,\omega,K,N}(\mathbf{k},\mathbf{p}), \tag{5.1}$$

where $a_1^{-1} = 1 - \frac{\lambda}{\pi}, \bar{a}_1^{-1} = 1 + \frac{\lambda}{\pi}$ and $R_{\varepsilon,\omega,K,N}(\mathbf{k},\mathbf{p})$ are the corrections (4.5) to the WI. We have seen that such corrections vanish removing cut-offs and *at fixed momenta*; however in (5.1) the corrections are integrated up to the cut-off scale, precisely where such corrections are not small, so that one is not legitimate to exchange the limit with the integrals. In order to bound (5.1) it is convenient to write it as

$$\mathcal{B}_{1,K,N}(\mathbf{k}) = \frac{\chi_N(\mathbf{k})}{D_\omega(\mathbf{k})} \sum_{\varepsilon=\pm} \frac{a_1 - \varepsilon \bar{a}_1}{2} \frac{\partial^2}{\partial h_{\mathbf{k},\omega} \partial \phi^+_{\mathbf{k},\omega}} \mathcal{W}_{\varepsilon,\omega,K,N}\bigg|_{h=\phi=0}, \tag{5.2}$$

where

$$\begin{aligned}
\mathrm{e}^{\mathcal{W}_{\varepsilon,\omega,K,N}(h,\phi)} = &\int P_Z(\mathrm{d}\psi^{(\leqslant N)}) \\
&\times \exp\left(\lambda Z^2 \int \frac{\mathrm{d}\mathbf{k}_1}{(2\pi)^2}\frac{\mathrm{d}\mathbf{k}_2}{(2\pi)^2}\frac{\mathrm{d}\mathbf{p}}{(2\pi)^2} v_K(\mathbf{p}) \big(\bar{\psi}^{(\leqslant N)}_{\mathbf{k}_1}\gamma_\mu \psi^{(\leqslant N)}_{\mathbf{k}_1-\mathbf{p}}\big)\big(\bar{\psi}^{(\leqslant N)}_{\mathbf{k}_2}\gamma_\mu \psi^{(\leqslant N)}_{\mathbf{k}_2+\mathbf{p}}\big)\right) \\
&\times \exp\left(T_0(\psi^{(\leqslant N)}, h) + T_1(\psi^{(\leqslant N)}, h) + \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \phi^{+(\leqslant N)}_{\mathbf{k},\omega} \psi_{\mathbf{k},\omega}\right),
\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}
T_0 &= Z \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} v_K(\mathbf{p}) h_{\mathbf{k},\omega}\psi_{\mathbf{k}-\mathbf{p},\omega} \frac{C_{\varepsilon\omega}(\mathbf{k}',\mathbf{k}'-\mathbf{p})}{D_{-\omega}(\mathbf{p})} \psi^+_{\varepsilon\omega,\mathbf{k}'}\psi_{\varepsilon\omega,\mathbf{k}'-\mathbf{p}} \\
T_1 &= Z\nu_- \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^2}\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} v_K(\mathbf{p}) h_{\mathbf{k},\omega}\psi_{\mathbf{k}-\mathbf{p},\omega} \frac{D_{-\varepsilon\omega}(\mathbf{p})}{D_{-\omega}(\mathbf{p})} \psi^+_{-\varepsilon\omega,\mathbf{k}'}\psi_{-\varepsilon\omega,\mathbf{k}'-\mathbf{p}}.
\end{aligned} \tag{5.4}$$

From (5.2) we see that $\mathcal{B}_{1,K,N}(\mathbf{k})$ is very similar to the two-point Schwinger function; the difference is that there is a new interaction $T_0 + T_1$ and that the external propagator is necessarily connected to this interaction.

Again the integration of the ultraviolet scales is done as in section 3, and after the integration of $N, N-1, \ldots, h+1$ the exponent in the functional integration is

$$\hat{\mathcal{V}}^{(h)}(\psi,h) = \mathcal{V}^{(h)}(\psi,0) + \sum_{\substack{n,m\\ n+m>0}} D^{(h)}_{m,2n-m}(\underline{\mathbf{x}},\underline{\mathbf{y}},\underline{\mathbf{z}}) \left[\prod_{i=1}^{n} \psi^{(\leqslant h)}_{\mathbf{x}_i}\right]\left[\prod_{i=1}^{n-m} \bar{\psi}^{(\leqslant h)}_{\mathbf{y}_i}\right]\left[\prod_{i=1}^{m} h_{\mathbf{z}_i}\right]. \tag{5.5}$$

We proceed as in section 3 analyzing in more detail the kernels $D^{(h)}_{1,1}$ and $D^{(h)}_{1,3}$ which have a non-negative dimension. There are several possible contributions. In the truncated expectations contributing to $D^{(h)}_{1,1}$ and $D^{(h)}_{1,3}$ there is necessarily a T_0 or T_1; we decompose $D^{(h)}_{1,2n-1}$ in an analogous way as in (4.10) writing, for $n = 2, 4$,

$$D^{(h)}_{1,2n-1} = D^{\alpha(h)}_{1,2n-1} + D^{\beta(h)}_{1,2n-1}, \tag{5.6}$$

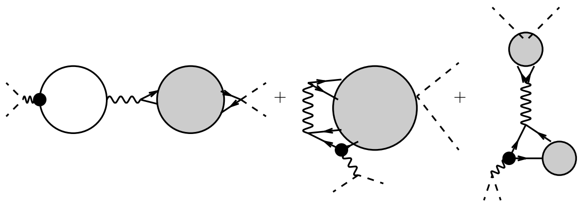

Figure 8. The contribution to $D^{\alpha(h)}_{3,1}$ obtained from T_0; the symbols are as in figure 6.

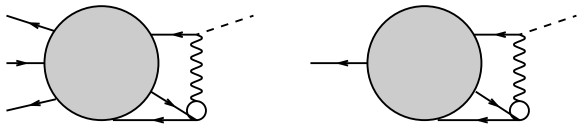

Figure 9. Structure of $D^{\beta(h)}_{3,1}$ and $D^{\beta(h)}_{1,1}$ are the symbols as in figure 6.

where in $D^{\alpha(h)}_{1,2n-1}$ are the terms such that the field $\psi_{\mathbf{k}-\mathbf{p},\omega}$ appearing in (5.4) is an external field (see figure 8), while in $D^{\beta(h)}_{1,2n-1}$ the field $\psi_{\mathbf{k}-\mathbf{p},\omega}$ is contracted. One immediately recognizes that

$$D^{\alpha(h)}_{1,3} = G^{(h)}_{2,1}, \tag{5.7}$$

where $G^{(h)}_{n,m}$ are the kernels appearing in (4.11), so that, from (4.20), for $h \geqslant K$,

$$\left\| D^{\alpha(h)}_{1,3} \right\| \leqslant C|\lambda|\gamma^{\frac{1}{2}(h-N)}. \tag{5.8}$$

On the other hand $D^{\beta(h)}_{n,1}$, $n = 1, 3$, has the structure shown in the following picture (in figure 9).

We can write

$$D^{\beta(h)}_{1,3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \int \lambda v_K(\mathbf{x}_1 - \mathbf{z}_1) g^{(h,N)}(\mathbf{x}_1 - \mathbf{z}_2) G^{(h)}_{4,1}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{z}_2; \mathbf{z}_1) \tag{5.9}$$

and by (4.20), for $h \geqslant K$

$$\left\| D^{\beta(h)}_{1,3} \right\| \leqslant C_1|\lambda|\|v\|_\infty \left\| G^{(h)}_{4,1} \right\| \cdot \sum_{j=h}^{N} \|g^{(j)}\|_1 \leqslant C_2|\lambda|\gamma^{2K}\gamma^{-2h}\gamma^{\frac{1}{2}(h-N)}. \tag{5.10}$$

We can proceed in the same way for $D^{\beta(h)}_{1,1}$ writing

$$D^{\beta(h)}_{1,1}(\mathbf{x}_1, \mathbf{x}_2) = \int \lambda v_K(\mathbf{x}_1 - \mathbf{z}_1) g^{(h,N)}(\mathbf{x}_1 - \mathbf{z}_2) G^{(h)}_{4,1}(\mathbf{x}_2, \mathbf{z}_2; \mathbf{z}_1) \tag{5.11}$$

and from (4.19)

$$\left\| D^{\beta(h)}_{1,1} \right\| \leqslant C_1|\lambda|\|v\|_\infty \left\| G^{(h)}_{1,1} \right\| \cdot \sum_{j=h}^{N} \|g^{(j)}\|_1 \leqslant C_2|\lambda|\gamma^{2K}\gamma^{-h}\gamma^{\frac{1}{2}(h-N)}. \tag{5.12}$$

From the above bounds it follows that the bound for $\mathcal{B}_{1,K,N}(\mathbf{k})$ is similar to that for the two-point Schwinger function up to an extra factor $\gamma^{\frac{1}{2}(h_{\mathbf{k}}-N)}$, if $h_{\mathbf{k}}$ is the scale of $\mathbf{k}$; hence it follows that $\mathcal{B}_{1,K,N}(\mathbf{k}) \to 0$ at $\mathbf{k}$ and K fixed, as $N \to \infty$.

In contrast in the case (2.4), (2.6), assuming that the bosonic cut-off is larger than the fermionic one $K > N$, there is no reason for which the terms in figure 9 vanish at $N \to \infty$, at

h fixed; indeed their local parts $\tilde{\lambda}_k$, $\tilde{z}_k$ have a non-trivial flow verifying $|\tilde{\lambda}_h - \alpha_\varepsilon \lambda_h| \leqslant C\lambda^2$ and $|\tilde{z}_h - \alpha_\varepsilon z_h| \leqslant C\lambda^2$, with $\alpha_- = O(\lambda)$ and $\alpha_+ = O(1)$, so that, by (3.23), they are non-vanishing and consequently the limit of $B_{2,K,N}$ as $N \to \infty$, $K \to \infty$ is also non-vanishing in this case, see [BFM].

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